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Linear System Stability via Liapunov's Direct Method and the Square Integral

LEE ROSENTHAL*
Hofstra University, Hempstead, N.Y.

Introduction

An application of Liapunov's second method to autonomous systems described by linear differential equations of the form

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = 0 \tag{1}$$

or by linear difference equations of the form

$$\sum_{i=0}^{n} a_i y(m+i) = 0 \qquad (m = \ldots -1, 0, 1, \ldots) \tag{2}$$

results in a set of necessary and sufficient conditions for asymptotic stability in the large.[1,2] These conditions must be equivalent, respectively, to the Routh stability criterion and the analogous criterion for discrete time systems, referred to as the table form criterion.[3–5] These equivalences have been shown by a number of authors.[6–11]

In this paper, the equivalence is established for both continuous and discrete time systems in completely parallel developments. This is accomplished by deriving and proving the stability criteria directly from appropriate Liapunov functions obtained in a very natural way.

The derivations begin with the square integral and summation defined, respectively, as

$$V = \int_t^\infty y^2(\tau)d\tau$$

for continuous time systems and by

$$V = \sum_{k=m}^{\infty} y^2(k)$$

crete time systems. The convolution integrals from transform theory along with residues are used to express the square integral and summation as quadratic forms in the state variables

$$V(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T\mathbf{B}\mathbf{Q}_n\mathbf{B}\mathbf{x}.$$

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* Assistant Professor of Engineering Science.

These forms are shown to be Liapunov functions for the systems.

The Routh criterion and the Routh-type criterion are then derived and proven to be necessary and sufficient conditions for the respective quadratic forms to be Liapunov functions and, therefore, for the systems to be asymptotically stable in the large.

Continuous Time Systems

A. *Derivation of the Liapunov function*

Consider the phase variable representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t), \quad y(t) = x_1(t) \tag{3}$$

of a system governed by the linear, constant coefficient differential equation given in Eq. (1).

Assume that the system is asymptotically stable in the large (ASL). Then, the square integral V defined by

$$V = \int_t^\infty x_1^2(\tau)d\tau$$

exists and can be evaluated using the convolution integral of the Laplace transform and residue theory (Ref. 12, Ref. 13, pp. 10–15) as the quadratic form

$$V(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T\mathbf{B}\mathbf{Q}_n\mathbf{B}\mathbf{x} \tag{4}$$

where (for n odd)

$$\mathbf{B} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ a_2 & a_3 & \ldots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_n & 0 & \ldots & 0 \end{bmatrix},$$

$$\mathbf{Q}_n = \begin{bmatrix} N_1 & 0 & N_2 & \ldots & N_{(n+1)/2} \\ 0 & -N_2 & 0 & \ldots & 0 \\ N_2 & 0 & N_3 & \ldots & N_{(n+3)/2} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ N_{(n+1)/2} & 0 & N_{(n+3)/2} & \ldots & N_n \end{bmatrix} \tag{5}$$

and the set of factors N_i is the solution of the set of equations

$$\begin{bmatrix} a_0 & a_2 & \ldots & a_{n-1} & 0 & \ldots & 0 \\ 0 & a_1 & \ldots & a_{n-2} & a_n & \ldots & 0 \\ 0 & a_0 & \ldots & a_{n-3} & a_{n-1} & \ldots & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & 0 & \ldots & a_0 & a_2 & \ldots & a_{n-1} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ \cdot \\ \cdot \\ \cdot \\ N_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1/a_n \end{bmatrix} \tag{6}$$

It is now shown that $V(\mathbf{x})$ is a Liapunov function for the system of Eq. (1). The integrand $(x_1)^2$ is non-negative. If $\mathbf{x} = \mathbf{0}$ at time t, it can be seen from Eq. (3) that x_1 and its first $n-1$ derivatives are zero at time t, and consequently that $x_1 = 0$ in the interval $[t, \infty]$. Therefore, $V = 0$. Also, if $V = 0$, then $x_1 = 0$ in that interval and $\mathbf{x} = \mathbf{0}$. It follows that V is positive definite and so is $V(\mathbf{x})$ as given in Eq. (4).

The derivative of V with respect to time given by

$$\dot{V} = d/dt\left[\int_t^\infty x_1^2(\tau)d\tau\right] = -x_1^2(t) \tag{7}$$

is negative semidefinite. Also, from the above, $\dot{V}$ is not identically zero anywhere other than $\mathbf{x} = \mathbf{0}$.

Using the linearity of the system with the above, it is seen that $V(\mathbf{x})$ is a Liapunov function for the system if it is ASL. Applying Liapunov's theorem, it is concluded that the system is ASL if and only if $V(\mathbf{x})$ is a Liapunov function.

B. Derivation and proof of the Routh criterion

The Routh stability criterion is now derived and proven directly from the above properties of $V(\mathbf{x})$ to be a necessary and sufficient condition for *ASL* for the system of Eq. (3).

Necessity

If $V(\mathbf{x})$ is a Liapunov function, then it must be positive definite. Since $a_n \neq 0$, the matrix $\mathbf{B}$ is nonsingular and it follows, therefore, that $\mathbf{Q}_n$ must be a positive definite matrix.

An iterative reduction procedure for the matrix $\mathbf{Q}_n$ and the set of Eqs. (6) produces the following set of relations between the principle minors of $\mathbf{Q}_n$ (Ref. 13, pp. 30–37).

$$|\mathbf{Q}_{n-i}| = \frac{1}{a_{n-i}^{(i-1)}\, a_{n-i-1}^{(i)}} |\mathbf{Q}_{n-i-1}| \quad (i = 0, 1, \ldots, n-1) \tag{8}$$

where

$$a_{n-i-1}^{(i)} = \frac{1}{a_{n-i}^{(i-1)}} \begin{vmatrix} a_{n-i}^{(i-1)} & a_{n-i-2}^{(i-1)} \\ a_{n-i+1}^{(i-1)} & a_{n-i-1}^{(i-1)} \end{vmatrix} \quad (i = 1, 2, \ldots, n-1) \tag{9}$$

and $a_i^{(0)} = a_i$

The matrix $\mathbf{Q}_n$ will be positive definite by Sylvester's criterion if and only if

$$|\mathbf{Q}_i| > 0 \quad (i = 1, 2, \ldots, n) \tag{10}$$

From Eqs. (8) and (10), the following set of inequalities must hold

$$a_n a_{n-1} > 0,\ a_{n-1}a_{n-2}^{(1)} > 0,\ \ldots,\ a_1^{(n-2)}a_0^{(n-1)} > 0$$

These are equivalent to the requirement that all of the terms

$$a_n,\ a_{n-1},\ a_{n-2}^{(1)},\ \ldots,\ a_0^{(n-1)}$$

have the same sign. Referring to Eq. (9), these terms are recognized as the Routh coefficients. Since the above requirement is a necessary condition that $V(\mathbf{x})$ be a Liapunov function, the Routh criterion is a necessary condition for *ASL* of the given system.

Sufficiency

It follows immediately from the above that if the Routh criterion is satisfied, then $V(\mathbf{x})$ is positive definite. Differentiating $V(\mathbf{x})$ with respect to time and substituting the system equation gives

$$\dot{V}(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T[\mathbf{A}^T\mathbf{B}\mathbf{Q}_n\mathbf{B} + \mathbf{B}\mathbf{Q}_n\mathbf{B}\mathbf{A}]\mathbf{x} \tag{11}$$

Using the easily verified relationship

$$\mathbf{BA} = \mathbf{A}^T\mathbf{B}$$

$\dot{V}(\mathbf{x})$ can be shown, with some matrix manipulation, to be

$$\dot{V}(\mathbf{x}) = -x_1^2(t) \tag{12}$$

which is identical to Eq. (7).

Therefore, if the Routh criterion is satisfied, $V(\mathbf{x})$ as given in Eq. (4) is a Liapunov function.

This result in conjunction with that under the previous subheading completes the derivation of the Routh criterion and the proof that it is a necessary and sufficient condition for the asymptotic stability in the large of the continuous time system of Eq. (1).

Discrete Time Systems

The derivation and proof of the analogous criterion for the discrete time case (called the table form) is completely parallel to the continuous time case. Therefore, only an outline of the results will be given (Ref. 13, pp. 5–9, 21–29).

A. Derivation of the Liapunov function

The phase variable representation of the difference equation given in Eq. (2) is

$$\mathbf{x}(k+1) = \mathbf{A}_0\mathbf{x}(k), \quad y(k) = x_1(k) \tag{13}$$

The square summation

$$V = \sum_{k=m}^{\infty} x_1^2(k)$$

can be evaluated as $V(\mathbf{x}) = \mathbf{x}^T\mathbf{B}\mathbf{Q}_n\mathbf{B}\mathbf{x}$, where $\mathbf{B}$ is given in Eq. (5), $\mathbf{Q}_n$ is given by

$$\mathbf{Q}_n = \begin{bmatrix} N_1 & N_2 & \ldots & N_n \\ N_2 & N_1 & \ldots & N_{n-1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ N_n & N_{n-1} & \ldots & N_1 \end{bmatrix}$$

and the factors N_i satisfy a set of equations analogous to Eq. (6).

B. Derivation and proof of the Routh-type criterion

Necessity

The matrix $\mathbf{Q}_n$ must be positive definite. The analogous sequence for the principle minors of $\mathbf{Q}_n$ is

$$|\mathbf{Q}_{n-i}| = -\frac{a_0^{(1)}a_0^{(2)}\ldots a_0^{(i)}}{a_0^{(i+1)}} |\mathbf{Q}_{n-i-1}| \quad (i = 0,1,\ldots,n-1)$$

where

$$a_0^{(i)} = \begin{vmatrix} a_0^{(i-1)} & a_{n-i+1}^{(i-1)} \\ a_{n-i+1}^{(i-1)} & a_0^{(i-1)} \end{vmatrix} \quad (i = 1, 2, \ldots, n) \tag{14}$$

The conditions that quarantee that $\mathbf{Q}_n$ is positive definite are

$$a_0^{(1)} < 0,\ a_0^{(2)} > 0,\ a_0^{(3)} > 0,\ \ldots,\ a_0^{(n)} > 0$$

Referring to Eq. (14), this set of inequalities is recognized as the table form stability criterion and had been shown to be a necessary condition for *ASL* (Ref. 4, pp. 97–98).

Sufficiency

If the table form criterion is satisfied, then $V(\mathbf{x})$ is positive definite. The forward difference $\Delta V(\mathbf{x})$ can be shown to be $\Delta V(\mathbf{x}) = -x_1^2$, which is negative definite, with the result that the table form criterion is sufficient to quarantee that $V(\mathbf{x})$ is a Liapunov function.

Therefore, the table form criterion is a necessary and sufficient condition for asymptotic stability in the large of the discrete time system of Eq. (2).

Conclusions

New Liapunov functions for the general nth order, autonomous, linear, continuous, and discrete time systems have been derived in a very natural way from the total square integral and summation.

The Routh and Routh-type stability criteria have then been derived and proven in a new way by completely parallel developments from the properties of these Liapunov functions.

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On the Singularity of Influence Coefficients of the Externally Pressurized Spherical Shells

TATSUZO KOGA*
National Aerospace Laboratory, Tokyo, Japan

1. Introduction

THE influence coefficients of pressurized spherical shells have been obtained by W. Nachbar,[1] G. B. Cline[2] and by D. Bushnell.[3] The numerical results for the externally pressurized spherical shells calculated by Bushnell show that the influence coefficients have singularity near one-half the classical buckling pressure, whereas the stiffness coefficients behave regular and nonzero. This indicates that the shell becomes softer as the external pressure increases and it cannot carry any load when the pressure reaches the value corresponding to the singularity. A question may then arise; why are the stiffness coefficients not zero? It is the intention of the present Note to clarify the question. It turns out that the singularity corresponds to the buckling at the edge of conical shells whose edge slides and rotates freely on the constraining surface of the conical shape. The conclusion may provide a supplementary explanation to the numerical result obtained recently by Baruch, Harari and Singer.[4]

2. Influence Coefficients

The basic nonlinear equations governing axisymmetric deformations of elastic thin shells of revolution have been formulated by E. Reissner.[5] The dependent variables may be represented as a result of the superposition of quantities belonging to the state of membrane stress and to that of bending stress. The equations are linearized with respect to the quantities belonging to the state of bending—taking the effect of the membrane stresses into account. Solution of the linearized equations yields the following expressions for the transverse shear resultant Q, the change of meridian tangent β, the horizontal component of stress resultant H, the meridional moment M_ϕ, and for the horizontal displacement u:

$$Q = (Eh)(R_1A + I_1B)\Psi \tag{1}$$

$$\beta = \lambda_0^2(\gamma_1A + \gamma_2B)\Psi \tag{2}$$

$$H = (Eh/\sin\phi)(S_1A + S_2B) \tag{3}$$

$$M_\phi = -(D/a)(S_3A + S_4B) \tag{4}$$

$$u = a\sin\phi[(S_1' - \cot\phi S_1)A + (S_2' - \cot\phi S_2)B] \tag{5}$$

where h is the thickness, a the radius, ϕ the meridional angle measured from the axis of rotation of the shell, E Young's modulus, D the bending rigidity of the shell, R_1 and I_1 are, respectively, the real and imaginary parts of the Bessel function of the first kind of order one, A and B are constants, and the prime indicates differentiation with respect to ϕ.

Here, the following abbreviations have been introduced:

$$\Psi = (\phi/\sin\phi)^{1/2} \tag{6}$$

$$\lambda_0 = [12(1 - \nu^2)]^{1/4}(a/h)^{1/2} \tag{7}$$

$$\gamma_1 = \rho R_1 + (1 - \rho^2)^{1/2}I_1 \tag{8a}$$

$$\gamma_2 = \rho I_1 - (1 - \rho^2)^{1/2}R_1 \tag{8b}$$

$$S_1 = -[(1 - 2\rho^2)R_1 - 2\rho(1 - \rho^2)^{1/2}I_1]\Psi \tag{9a}$$

$$S_2 = -[(1 - 2\rho^2)I_1 + 2\rho(1 - \rho^2)^{1/2}R_1]\Psi \tag{9b}$$

$$S_3 = \lambda_0^2\{\rho[(R_1\Psi)' + \nu\cot\phi(R_1\Psi)] + (1 - \rho^2)^{1/2}[(I_1\Psi)' + \nu\cot\phi(I_1\Psi)]\} \tag{9c}$$

$$S_4 = \lambda_0^2\{\rho[(I_1\Psi)' + \nu\cot\phi(I_1\Psi)] - (1 - \rho^2)^{1/2}[(R_1\Psi)' + \nu\cot\phi(R_1\Psi)]\} \tag{9d}$$

where ρ is the ratio of the applied pressure to the classical buckling pressure of complete spherical shells, and ν is Poisson's ratio.

It should be noted that the approximation

$$V = 0 \tag{10}$$

has been made in the derivation of the aforementioned expressions.

The influence coefficients are now obtained by prescribing the inhomogeneous boundary conditions, $H = H_e$ and $M_\phi = M_{\phi e}$, where the subscript e indicates the values at the edge

$$u_e = C_{11}H_e + C_{12}M_{\phi e}, \quad \beta_e = C_{21}H_e + C_{22}M_{\phi e} \tag{11}$$

The result is identica lto that of Bushnell[3] and it may be written in the form

$$\begin{aligned} C_{11} &= (a\sin^2\alpha/Eh)(1/\Delta)(S_1'S_4 - S_2'S_3 - \Delta\nu\cot\alpha) \\ C_{12} &= (\lambda_0^4\sin\alpha/Eh)(1/\Delta)(S_1'S_2 - S_2'S_1) \\ C_{21} &= (\lambda_0^2\sin\alpha/Eh)(1/\Delta)(\gamma_1S_4 - \gamma_2S_3) \\ C_{22} &= (\lambda_0^6/Eh\alpha)(1/\Delta)(\gamma_1S_2 - \gamma_2S_1) \end{aligned} \tag{12}$$

where

$$\Delta = S_1S_4 - S_2S_3 \tag{13}$$

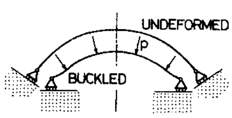

Fig. 1 Boundary condition $H_e = M_{\phi e} = 0$.

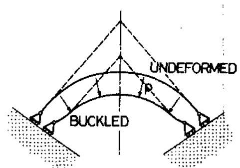

Fig. 2 Boundary condition $Q_{\alpha e} = M_{\phi e} = 0$.

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* Research Engineer.